

Matching Market Design with Constraints

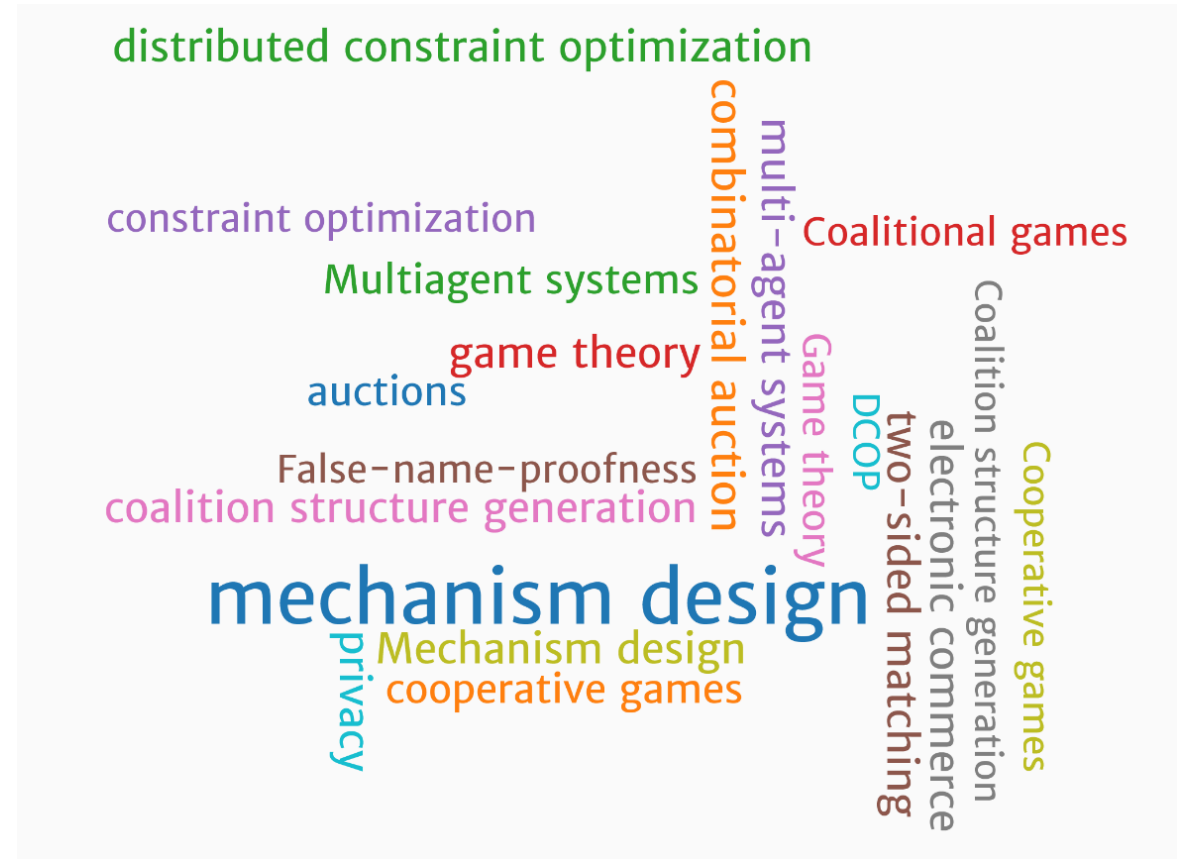
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九州大学

Yokoo's Research Topics

- Computer Science
 - Artificial Intelligence
 - Multi-agent Systems
- Multi-agent systems: systems composed of multiple interacting computing elements called agents
 - Each agent might have its own goal.
 - Game theory is commonly used as a theoretical foundation of multi-agent systems.
 - Deciding a collective decision making rule among agents = mechanism/market design

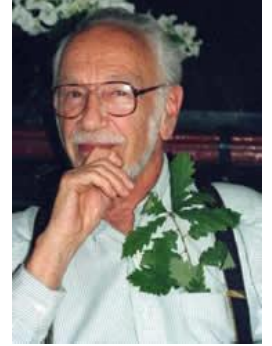


About this talk

- I'll present a simple and general abstract model of constrained two-sided matching, and introduce a few representative constraints that can be formalized using this model.
- I'll also present general mechanisms that work with two general classes of constraints.

Two-sided matching

- The goal is to find a desirable combination of two parties: students $\Leftrightarrow$ laboratories or colleges, workers $\Leftrightarrow$ companies, medical residents $\Leftrightarrow$ hospitals
- Beautiful theoretical results have been obtained; the Deferred Acceptance (DA) mechanism (aka Gale-Shapley mechanism) is strategyproof (for students), and obtains the student optimal matching among all stable matchings.
- However, these results are applicable only for the standard model, where only distributional constraints are the maximum quota (capacity limit) of each college.



Why constraints matter?

- It is common that social requirements impose various distributional constraints on the outcome of the matching.
 - A college needs a certain number of students to operate.
 - Some medical residents must be assigned to a rural hospital.
- Is it really reasonable to assume each college has a strict, exogenously given maximal quota?
 - We usually have some flexibility (e.g., allocating more human resources to popular colleges).
 - Such flexibility can be naturally represented by constraints!

Outline

- Introduction
- General abstract model
- Instances of distributional constraints
 - Regional maximum quotas
 - Minimum quotas
- Intermission: why I became interested in this topic
- Hereditary M $\sqcap$ -convex set constraints
- Hereditary constraints
- Tradeoff between fairness and efficiency

General abstract model (I)

- A set of n students $S = \{s_1, s_2, \dots, s_n\}$
- A set of m colleges $C = \{c_1, c_2, \dots, c_m\}$
- A set of contracts X .

A contract $(s, c) \in X$ represents the fact that s is matched to c .

Notation: for $Y \subseteq X$, $Y_s = \{(s, c) \in Y \mid c \in C\}$, $Y_c = \{(s, c) \in Y \mid s \in S\}$.

- Each student has a strict preference $\succ_s$ over $X_s \cup \{\emptyset\}$.
- Each college has a strict preference $\succ_c$ over X_c .

Note: all contracts in X are acceptable for colleges.

- Y is a matching if $|Y_s| \leq 1$ and $Y_s \succ_s \emptyset$ holds for all $s \in S$.

Note: I abuse $\succ_s$ (and $\succ_c$), e.g., $Y_s \succ_s c$.

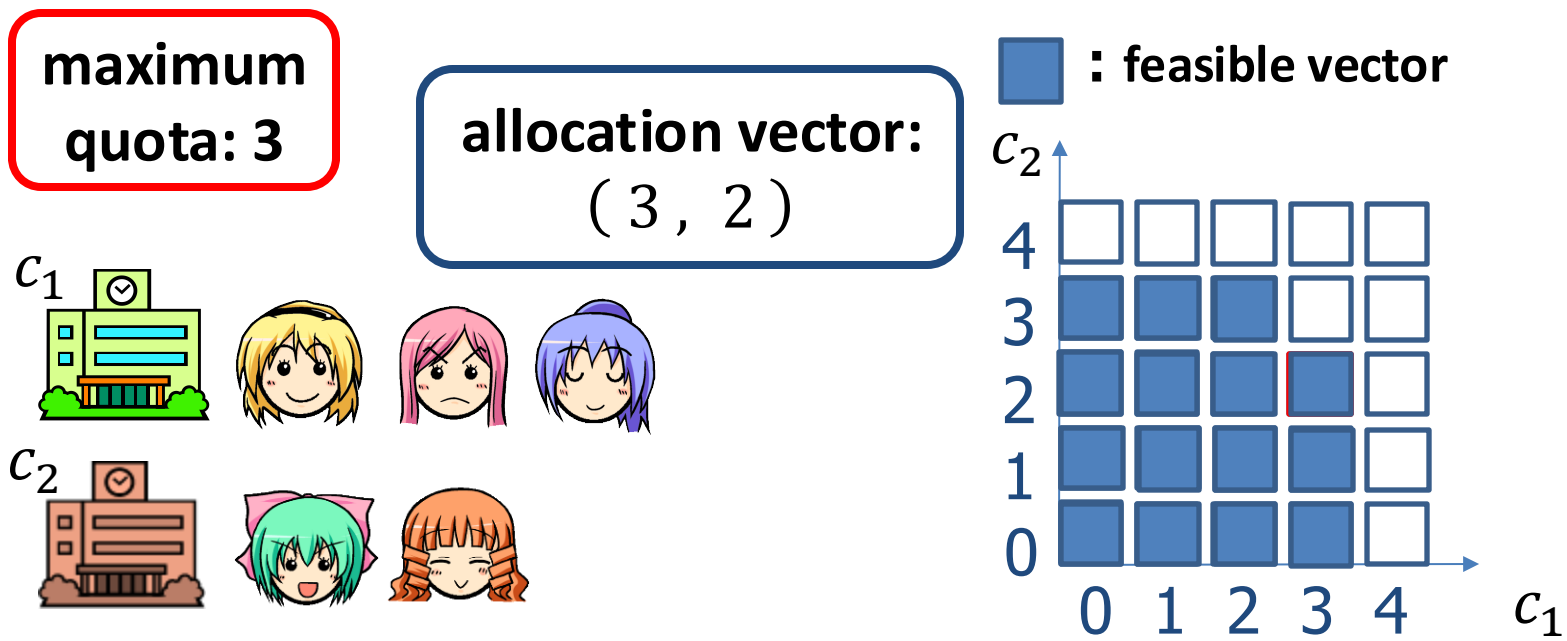
General abstract model (II)

Notation: for $Y \subseteq X$ and $x \in X$, $Y + x$ denotes $Y \cup \{x\}$ and $Y - x$ denotes $Y \setminus \{x\}$.

- For $Y \subseteq X$, $v(Y)$ is an allocation vector that describes the number of students allocated to each college:
 $(|Y_{c_1}|, |Y_{c_2}|, \dots, |Y_{c_m}|)$.
- $f: \mathbb{Z}^m \rightarrow \{0, -\infty\}$ represents distributional constraints:
matching Y is feasible if $f(v(Y)) = 0$.

Standard model

- Each college c has its maximum quota q_c .
- For $v = (v_{c_1}, \dots, v_{c_m})$, $f(v) = 0$ iff $\sum_{c \in \mathcal{C}} v_c \leq n$, and $v_c \leq q_c$ for each $c \in \mathcal{C}$.



Desirable properties

Fairness: Feasible matching Y is fair if no student has justified envy; student s has justified envy toward another student s' if $(s', c) \in Y$, $c \succ_s Y_s$, and $s \succ_c s'$ hold.

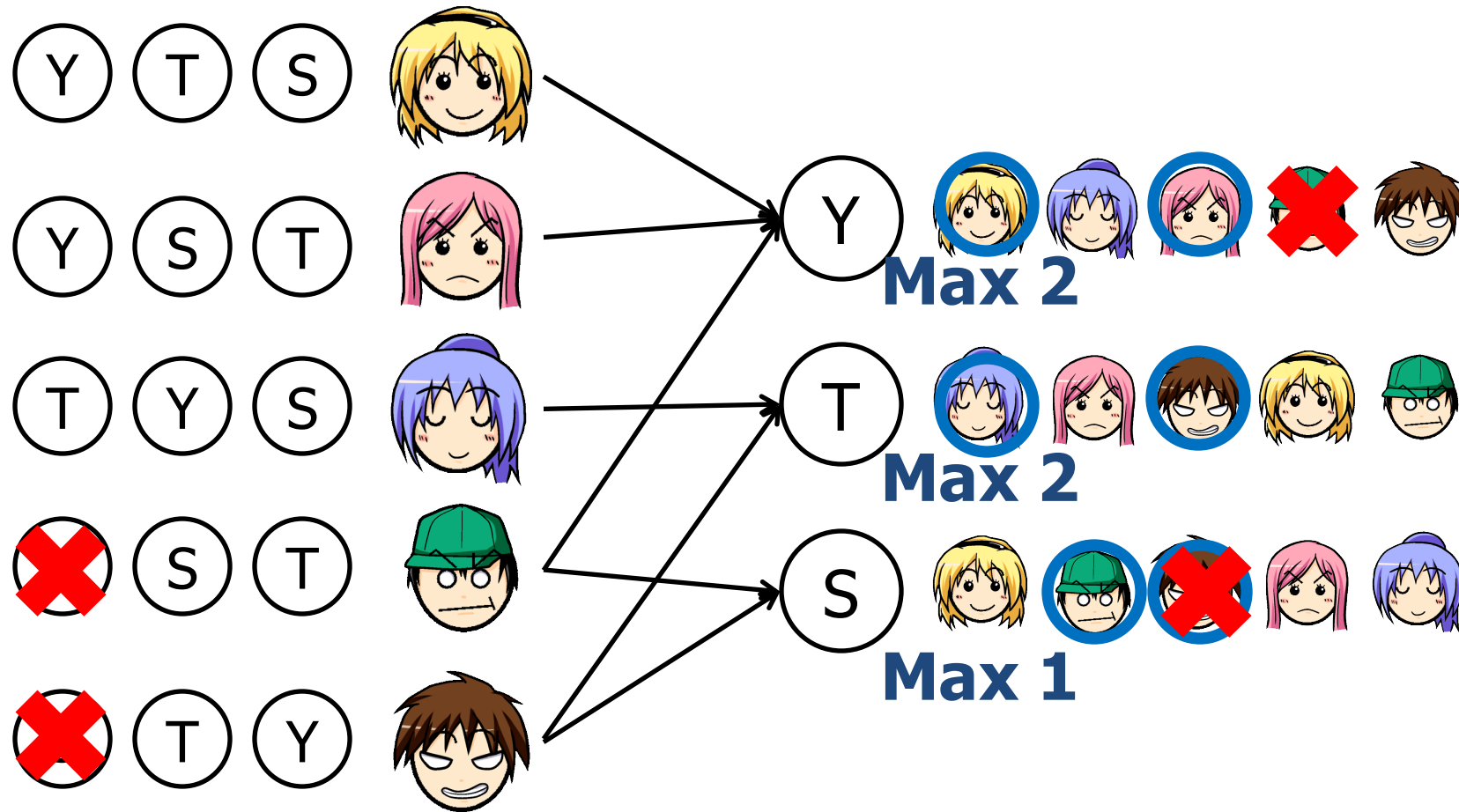
Nonwastefulness: Feasible Matching Y is nonwasteful if no student claims an empty seat; student s claims an empty seat of c , if $c \succ_s Y_s$ and $f(v(Y \setminus Y_s + (s, c))) = 0$.

Fairness + nonwastefulness = (standard) **stability**.

DA mechanism (Gale & Shapley, American Mathematical Monthly, 1962)

- In k -th round :
 - Each student applies to her most preferred college from which she has not rejected so far.
 - Each college deferred accepts applying students based on its preference up to its maximum quota (note that the acceptance is only tentative).
 - The rest of students are rejected.
- The rejected students apply to their next colleges. Each college deferred accepts applying students (there is no distinction between students who are deferred accepted in the previous round and newly applied students) up to its maximum quota. Repeat these rounds until all students are deferred accepted; finalize the current tentative assignment.

Application of DA mechanism



Characteristics of DA mechanism

- **Strategyproofness**: for each student, declaring her true preference is optimal regardless of the actions of other students.
 - She cannot obtain a better outcome by modifying her preference.
- The obtained matching is **fair** and **nonwasteful**. Moreover, it is the **optimal** matching for students within all stable matchings (within all stable matchings, all students prefers the matching obtained by the DA mechanism, including ties).

So far, the DA is great, but it fails to satisfy distributional constraints...

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Regional maximum quotas (Kamada & Kojima, AER-2015)

In addition to the standard model,

- A set of regions $R = \{r_1, r_2, \dots\}$, each region r is a subset of colleges. Regions are mutually disjoint.
- Each region has its own (regional) maximum quota q_r .
- $f(v) = 0$ only if for each $r \in R$, $\sum_{c \in r} v_c \leq q_r$ holds.

Motivation:

- Applying to Japanese resident matching. By reducing the number of residents in metropolitan areas like Tokyo, enough residents are assigned to rural areas.
- Currently, Artificial Cap approach is used, i.e., the individual maximum quotas are set artificially small to satisfy regional quotas, but this approach is inflexible.

Minimum quotas

(Fragiadakis, Y. *et al.* ACM-TEAC-2015)

In addition to the standard model,

- Each college has its own minimum quota p_c .
- For $v = (v_{c_1}, \dots, v_{c_m})$, $f(v) = 0$ iff $\sum_{c \in C} v_c \leq n$, and for each $c \in C$, $p_c \leq v_c \leq q_c$ holds.
- We assume all students are acceptable for all colleges and vice versa. We also assume $\sum_{c \in C} p_c \leq n$.

Motivation:

- Some medical residents must be allocated to a rural hospital.

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How did I become interested in this topic (I)

- Although I had basic knowledge on the DA mechanism as a subfield of market/mechanism design, I never imagined that I'm going to work in this topic by myself.
- The original paper by Gale & Shapley was published in 1962 (the year I was born).
- I did not think any important research topic remains in this subfield.

How did I become interested in this topic (II)

- I was in charge of assigning students to laboratories for their graduate thesis in Kyushu University, Electrical Engineering and Information Science Department in 2011.
- So far, a mechanism called Boston mechanism was used.
 - Each student declares her most desired laboratory, the second-best, and so on.
 1. Students are assigned to their first choice laboratories. If a laboratory is full to its maximum quota, students are selected by a common ordering (GPA) up to the maximum quota and the rest of students are rejected.
 2. For the rejected students and laboratories that are not filled, matching is continued base on the second-best choices, and so on.

What's wrong with the Boston mechanism? (I)

- Students need to act strategically.
 - She must choose her choices wisely, considering the preferences of students who have better GPA than herself.
- If students fail to predict the actions of other students, a socially undesirable outcome might occur.

What's wrong with the Boston mechanism? (II)

Example:

- Student 1 wants to join laboratory A. However, laboratory A seems to be very popular and student 1 is not confident she can be accepted or not. If she fails, she might end up to be assigned to a very unpopular laboratory.
- Thus, she chooses laboratory B as her first choice to avoid the risk.
- However, most of students also try to avoid the risk and laboratory A turns out to be not very popular, and student 2, whose GPA is much lower than student A, is accepted to laboratory A.
- This outcome is undesirable since student 1 has justified envy toward student 2.

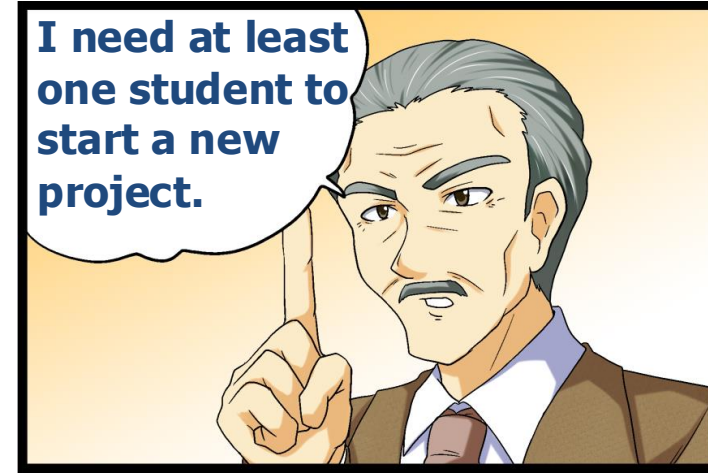
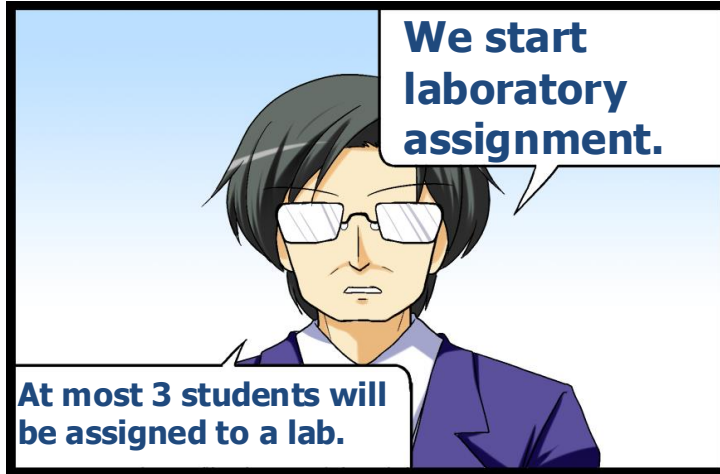
How did I become interested in this topic (III)

- I proposed to apply the DA mechanism at the faculty meeting and accepted.
- Previously, the number of students assigned to each faculty member was fixed (three students for each full professor, and two students for each associate professor).
- However, in this year, the number of students was low.
- Thus, we decided that at least two students are assigned to each full professor, and one student is assigned to each associate professor, and optionally one more student can be assigned according to their preferences.
- By this change, strategic decision making by students becomes too complicated if we apply the Boston mechanism.

How did I become interested in this topic (IV)

- Initially, I thought I can simply apply the standard DA mechanism.
- However, we need to satisfy minimum quotas (two for full professors and one for associate professors).
- I thought there must be some existing works on such a simple, fundamental problem and ask a professional (Prof. Fuhito Kojima).
- However, to my surprise, he said that this problem is very difficult and impossible to solve; no stable matching exists in general (but I already stated that the celebrated DA mechanism would work fine at the faculty meeting).

The Melancholy of Prof. Y (Part I)



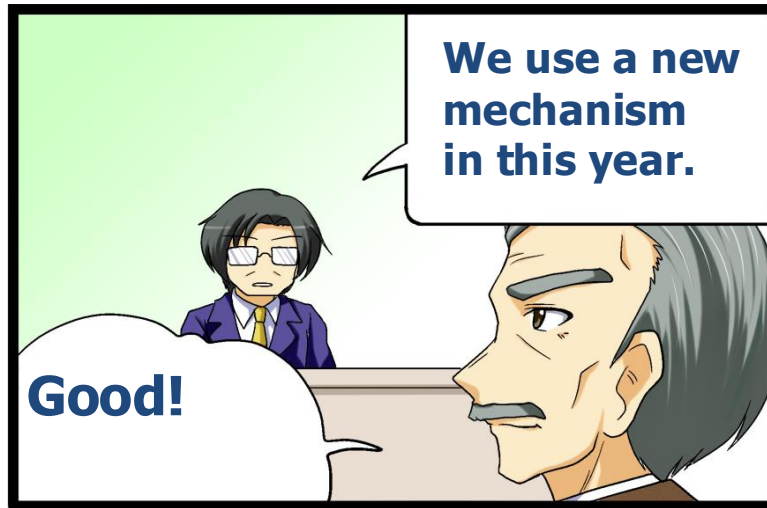
Prof. Y	0	



How did I become interested in this topic (V)

- The laboratory assignment problem at Kyushu University is based on the common ordering (GPA), each laboratory cannot use its own preference. The common ordering is called Master List (ML).
- By using this fact, I managed to create a mechanism that satisfies minimum quotas (one instance of serial dictatorship mechanisms) while guaranteeing several desirable properties (e.g., strategyproofness).

The Melancholy of Prof. Y (Part II)



A comic panel showing a table and a list of students. The table has four rows and three columns. The first column contains "..." in the first three rows and "Prof. Y" in the fourth row. The second column contains the numbers 3, 1, 2, and 1. The third column contains the heads of the students: Alice (orange hair), Becky (green hair), Carol (pink hair), and Alice (orange hair). Below the table is a list of all seven students' heads, with the letters "ML" to the left.

...	3	Alice, Becky, Carol
...	1	Becky
Prof. Y	2	Carol, Alice
...	1	Alice

ML
Alice, Becky, Carol, Alice, Becky, Carol, Alice



How did I become interested in this topic (VI)

- By further modifying the mechanism, we can utilize the preference of each laboratory (by using GPA/ML as a tie-breaker).
- Also, I managed to create a mechanism that does not rely on ML at all.
- Actually, many research issues remain when some constraints are imposed on a feasible matching.
- I have been working on constrained matching problems, in which various distributional constraints (e.g., minimum quotas, regional maximum quotas, affirmative action) are imposed.

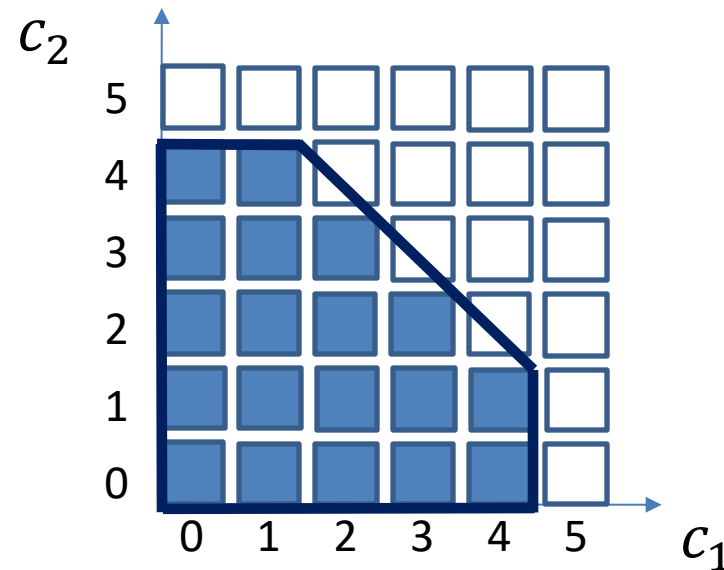
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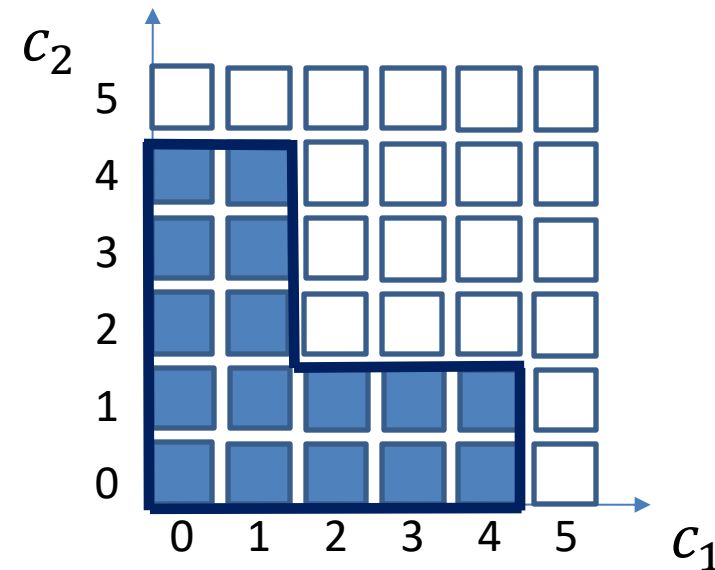
Distributional constraints

- **Heredity**: if vector v is feasible, then any vector smaller than v (by element-wise comparison) is also feasible.
- M $\sqsubseteq$ -convex set: no hollow ($\sqsubseteq$: natural)

hereditary and M $\sqsubseteq$ -convex set



hereditary,
not M $\sqsubseteq$ -convex set



$M \bowtie$ -convex set

(Murota, MathProgramming-1998)

We say a set of m -element vectors F is an $M \bowtie$ -convex set if for all $v, v' \in F$ and i where $v_i > v'_i$, the following condition holds:

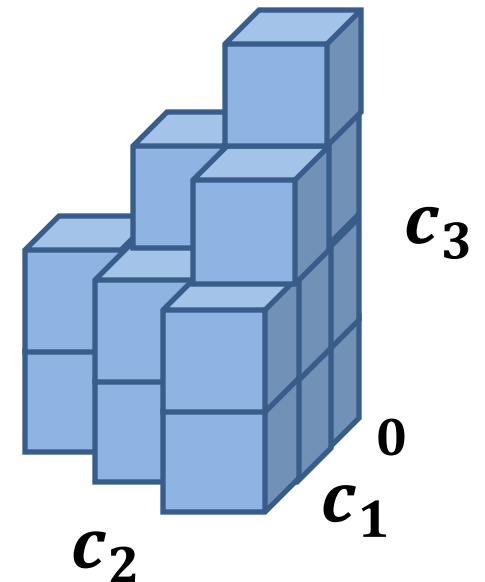
- There exists $j \in \{0\} \cup \{k \in \{1, 2, \dots, m\} \mid v_k < v'_k\}$ such that
 $v - e_i + e_j \in F$ and $v' + e_i - e_j \in F$ hold.

Here, e_i is an m -element vector where its i -th element is 1 and other elements are 0, and e_0 is an m -element 0-vector.

Constraints represented as M $\sqcap$ -convex set (I)

- It is easy to verify that f satisfies heredity and M $\sqcap$ -convexity conditions in the standard model and the regional maximal quota model.
 - Feasible vectors form a hyper-rectangle, trimmed by hyper-planes.

$$n = 3, q_{c_1} = q_{c_2} = q_{c_3} = 3, \\ r = \{c_1, c_2\}, q_r = 2.$$



Constraints represented as M -convex set (II)

Minimum quotas:

- Clearly, minimum quotas do not satisfy heredity condition: $(0, 0, \dots, 0)$ is infeasible.
- Let us consider *semi-feasible* vectors: any vector that is smaller than (or equal to) any feasible vector is semi-feasible.
- Then, semi-feasible vectors form an M -convex set (similar to regional maximum quotas).
- Since we assume all students are acceptable for all colleges and vice versa, as long as a mechanism returns a matching where each student is matched to some college, it is feasible.

Other Constraints represented as M -convex set constraints

- Hierarchical regional maximum/minimum quotas
- Type-specific maximum/minimum quotas
- Distance constraints
- ...

Mechanism for $M \sqsupset$ -convex set: Generalized DA

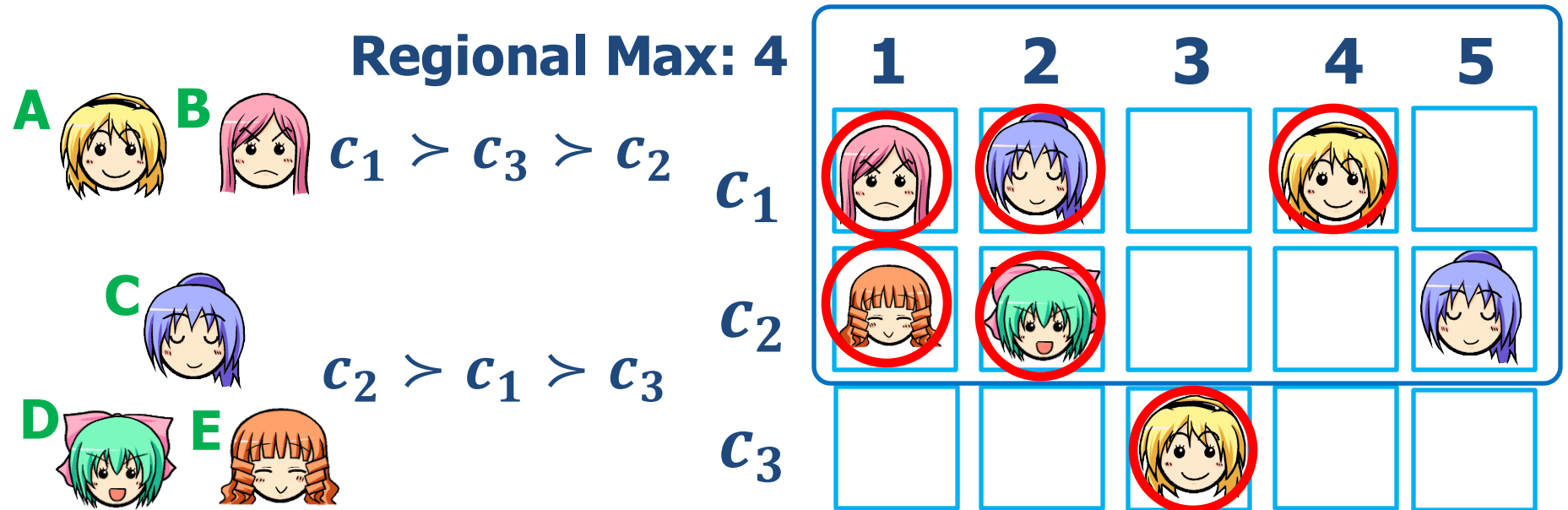
- To run DA with distributional constraints, we need more ingredients in addition to $\succ_c$.
- In the regional maximum quota model, where $r = \{c_1, c_2\}$, $q_r = 1$, if s_1 applies to c_1 , and s_2 applies to c_2 , the mechanism must choose one of them (i.e., one contract from $\{(s_1, c_1), (s_2, c_2)\}$) to satisfy feasibility.
- Let us assume each contract x has its unique weight $w(x)$, which represents its relative importance.
- The colleges as a whole prefer (s_1, c_1) over (s_2, c_2) if $w((s_1, c_1)) > w((s_2, c_2))$ holds.
- We assume weights respect college's preference, i.e., if college c prefers s over s' , $w((s, c)) > w((s', c))$ holds.

Details of Generalized DA

1. $Re \leftarrow \emptyset$.
 2. $Y \leftarrow Ch_S(X \setminus Re), Y' \leftarrow Ch_C(Y)$.
 3. If $Y = Y'$ then return Y ,
otherwise $Re \leftarrow Re \cup (Y \setminus Y')$, go to 2.
- Re : rejected set, which represents a set of contracts that are proposed by students and rejected by colleges. Students are not allowed to propose contracts in Re .
 - Initially, Re is empty. Thus, students can choose their most preferred contracts and proposes them to colleges. This set is Y .
 - Then, colleges choose the most preferred (and feasible) subset Y' from Y .
 - If no contract is rejected, the mechanism terminates.
 - Otherwise, the rejected contracts are added to Re , and the mechanism repeats the same procedure.
 - Ch_C can be computed by a greedy algorithm for $M^\natural$ -convex set constraints.

Example: Generalized DA

- c_1 and c_2 belong to the same region with regional maximum quota 4.
- Assume a contract with a higher ranked student in each college has a larger weight; for students with the same rank, prefer c_1 over c_2 .
- The obtained matching is fair, but can be wasteful.



Characteristics of Generalized DA

Theorem (Kojima, Tamura & Y., JET-2018):

Assume f (distributional constraints) is represented as a hereditary M $\sqcap$ -convex set,

Then Generalized DA is:

- (i) strategyproof for students,
- (ii) obtains the student-optimal matching among all Hatfield & Milgrom-stable matchings (which implies fairness), and
- (iii) its time complexity is $O(|X|^2)$.

Hatfield-Milgrom (HM)-stability

(Hatfield & Milgrom, AER-2005)

We say matching Y is HM-stable if

1. $Y = Ch_C(Y) = Ch_S(Y)$, and
2. there exists no $x \in X \setminus Y$ s.t.
 $x \in Ch_C(Y + x)$ and $x \in Ch_S(Y + x)$ hold.
 - Intuitively, HM-stability requires that there exists no mutually more preferred contract than Y .
 - Assuming weights respect $\succ_C$ (when $s \succ_C s'$, $w((s, c)) > w((s', c))$ holds), HM-stability implies fairness; .

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Handling more general constraints

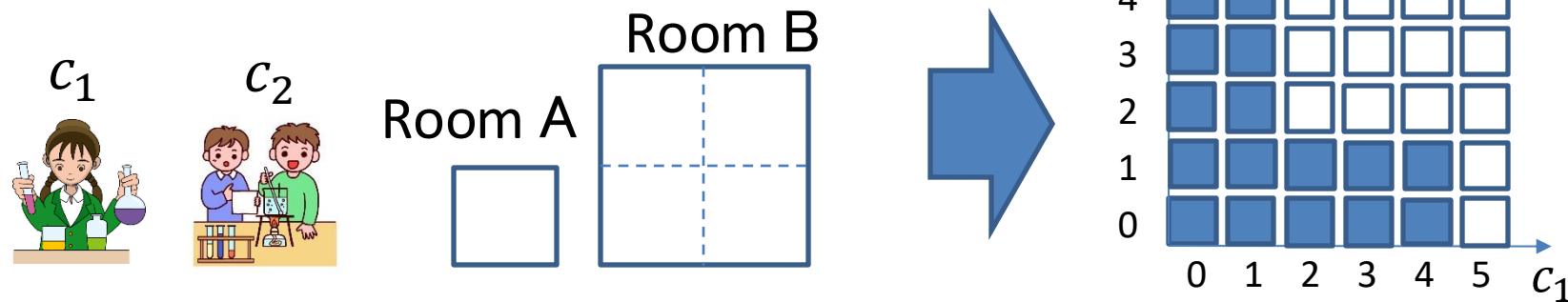
Question: Is it possible to handle any broader class of constraints?

Answer: Yes, hereditary constraints can be handled by Adaptive DA (Goto, Kojima, Kurata, Tamura & Y., American Economic Journal: Microeconomics, 2016)

Constraints violating M $\bowtie$ -convexity

Project-student-room assignment problem

- assign students and a room to each project
- the maximum quota of a project is determined by the size of the allocated room
- This problem is a special case of a project-student-resource allocation problem.

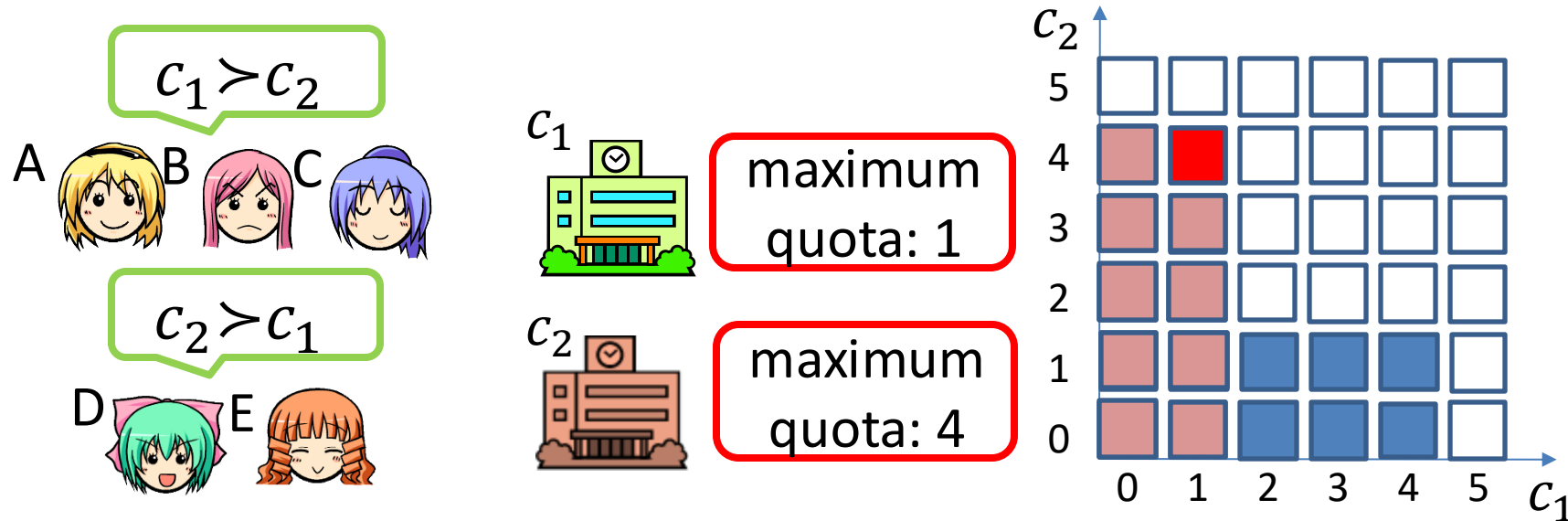


Mechanisms for hereditary constraints

- Fairness and nonwastefulness are incompatible in general.
- If we give up fairness or nonwastefulness completely, we can use the following trivial strategyproof mechanisms:
 - Artificial Cap DA (ACDA) --- fair
 - Serial Dictatorship (SD) --- nonwasteful

Artificial Cap DA (ACDA)

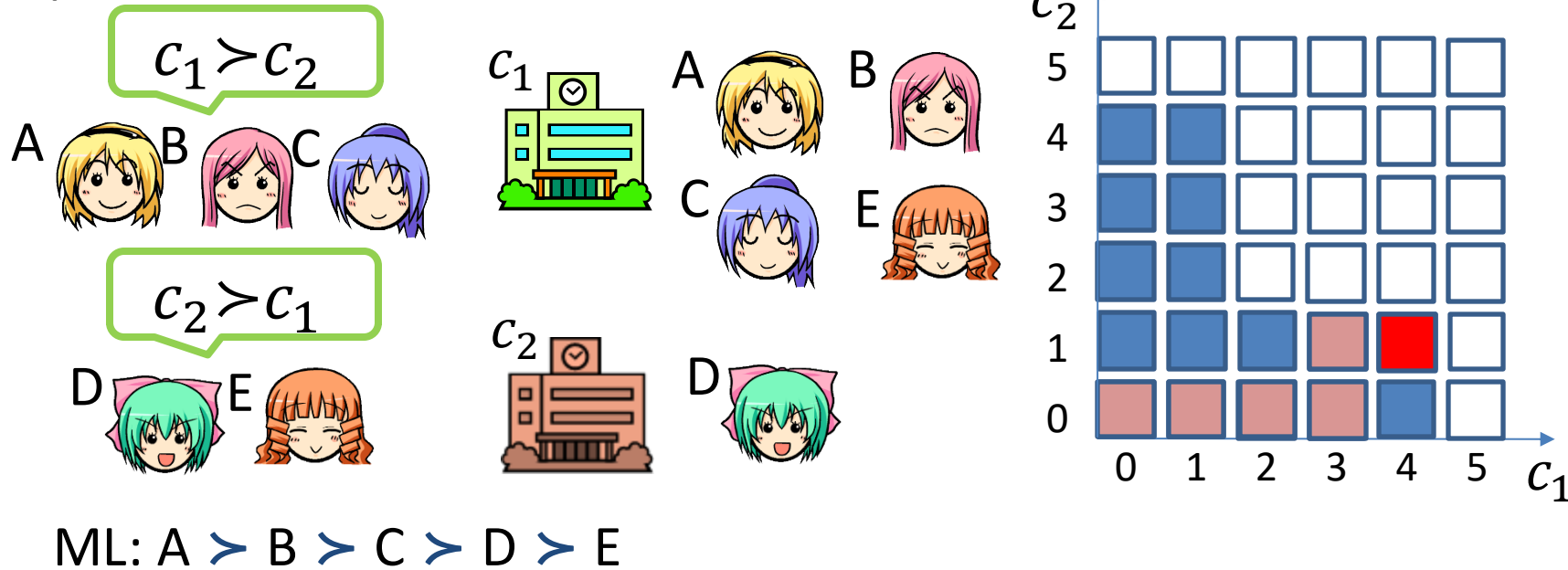
- Choose one maximal feasible vector.
- Use the value of each element as the maximum quota of the corresponding college, and run the standard DA.



- To guarantee strategyproofness, the maximal vector must be chosen exogenously.
- It is fair but too wasteful.

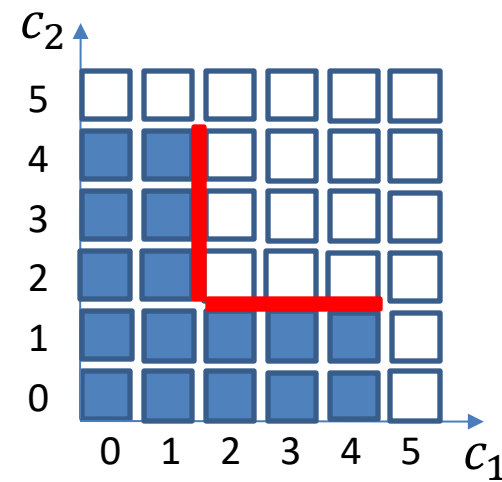
Serial Dictatorship (SD)

- Assume a common ordering among students (Master List, ML) exists.
- Assign students sequentially according to ML; a student can choose her most preferred college as long as constraints are satisfied.
- It is nonwasteful (and Pareto efficient), but it is too unfair (it completely ignores colleges' preferences).



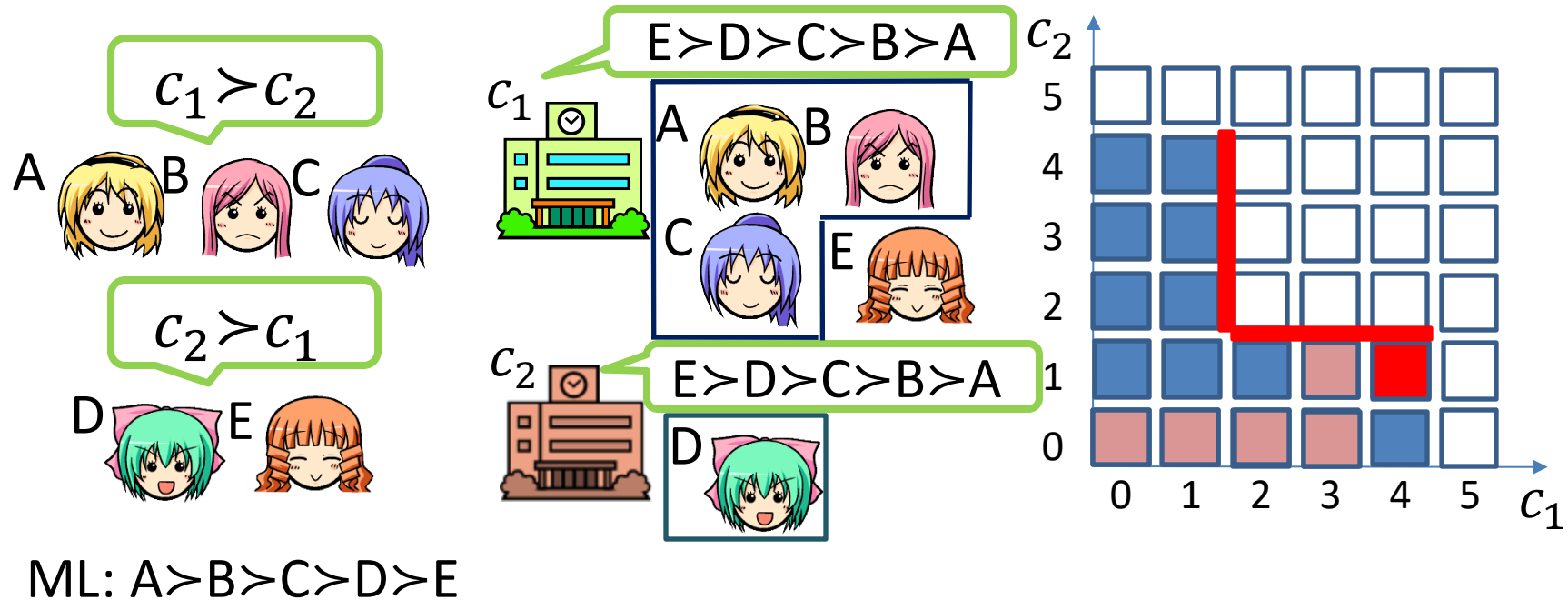
Adaptive DA

- It uses both ML and colleges' preferences; guarantees nonwastefulness, but compromises on fairness.
- We assume each college c has its own maximum quota q_c .
- There are multiple stages.
- In each stage, a new student is added according to ML.
- The standard DA is applied to the current students.
- If a certain condition is satisfied, finalize the current matching, revise maximum quotas, and move to the next stage.
- Otherwise, reset the current matching; move to the next stage.
- Condition: whether the current matching **reaches the wall**, which represents distributional constraints (no more student can be assigned to college c even though q_c is not reached yet).



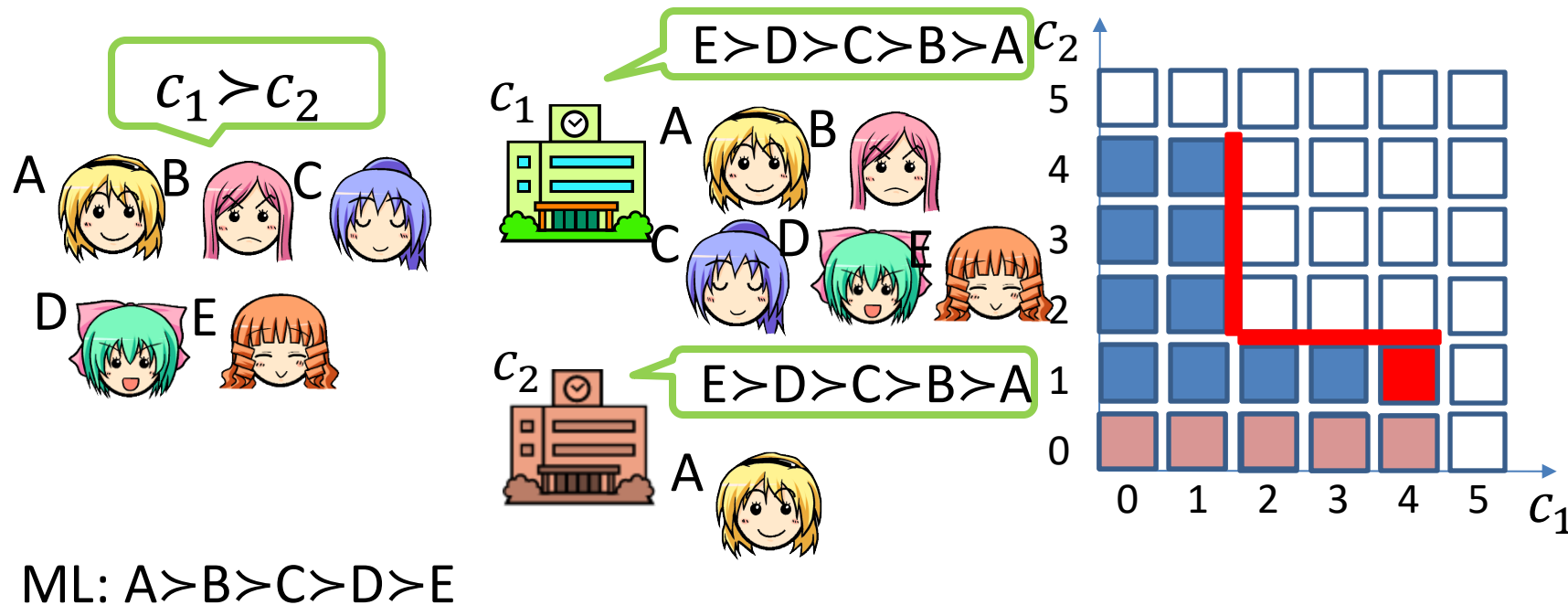
Example (Adaptive DA)

- The resulting maximal vector tends to be better for students compared to the ACDA (it reflects students' preferences).



Example (Adaptive DA)

- Students can compete based on colleges' preferences; the Adaptive DA can be more fair than the SD.



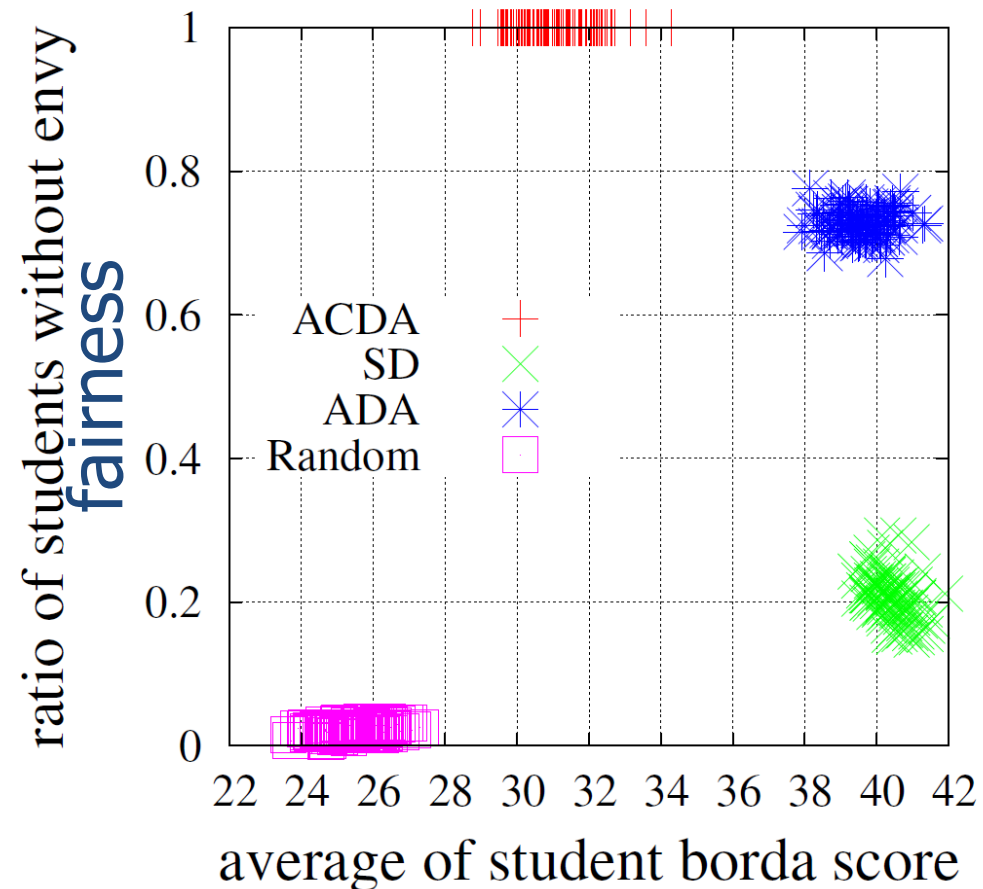
Characteristics of Adaptive DA (I)

Nonwastefulness:

- If student s is not allocated to a course, then s is either not full or reaching the wall; moreover, s is not a constraint.

Pareto frontier:

- There exists a tradeoff between efficiency and fairness (students' welfare).
- To improve fairness of the outcome, we need to assign at least one student to a course.



efficiency

Characteristics of Adaptive DA (II)

Strategyproofness:

- Not trivial: student s can lie in order to reach the wall earlier so that the current matching is fixed; s can secure a seat of college c (which might not be available in the later stage).
- However, we can prove this manipulation is meaningless.
 - If you can reach a wall now and secure the seat of college c , then you can reach the wall in the next stage, or somebody else reaches the wall (and you can secure a seat better than c).

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Tradeoff between fairness and efficiency (I)

Let us introduce a weaker notion of nonwastefulness.

Nonwastefulness: Feasible Matching Y is nonwasteful if no student claims an empty seat; student s claims an empty seat of c , if $c \succ_s Y_s$ and $f(v(Y \setminus Y_s + (s, c))) = 0$.

Weak Nonwastefulness: Feasible Matching Y is weakly nonwasteful if no student strongly claims an empty seat; student s strongly claims an empty seat of c , if $c \succ_s Y_s$ and $f(v(Y + (s, c))) = 0$.

Tradeoff between fairness and efficiency (II)

Existence of any fair and strategyproof mechanism

	Maximum quotas only	M ^h -convex set	Hereditary constraints
Pareto efficiency	X	X	X
Nonwastefulness	○	X	X
Weak Nonwastefulness	○	○	X

X : hold even without strategyproofness

X : require strategyproofness

(Cho, Y., et al. Economics Letters-2022)

Tradeoff between fairness and efficiency (III)

- We are expanding/generalizing constraints s.t. students are assigned to colleges in a more flexible way, in the hope that students' welfare would be better.
- However, if we require strict fairness, we cannot guarantee a very weak requirement of efficiency/students' welfare!
- I believe we need a weaker notion of fairness, s.t. it becomes compatible with some reasonable efficiency notion (as well as strategyproofness).

Idea: Envy-freeness up to k peers (EF- k , Cho, Y., et al. AAMAS-2024)

- For matching Y and student s ,
Let $E(Y, s)$ denote $\{s' \mid s \text{ has justified envy toward } s' \text{ in } Y\}$
- We say Y is envy-free up to k peers, if $\forall s \in S, |E(Y, s)| \leq k$ holds.
- EF-0 is equivalent to the standard envy-freeness/fairness.
- Any matching is EF- $(n - 1)$ fair (where n is the number of students).

New strategyproof mechanisms for hereditary constraints

SD with an optimal master list (SD*)

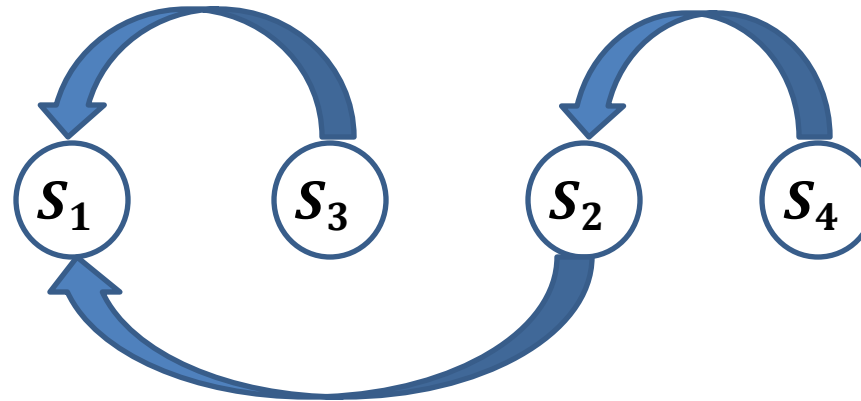
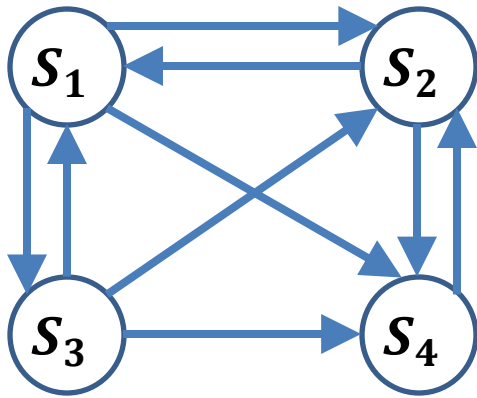
- Choose an optimal master-list (based on colleges' preferences) such that (worst-case) k is minimized and run SD.
- Pareto efficient, can be EF- k , where k is determined by colleges' preferences; k can be $n - 1$, but it is much smaller in practice assuming colleges' preferences are similar.

Sample and Deferred Acceptance (SDA)

- Choose k sample students (they are assigned by SD)
- Fix the maximum quota of each college based on sample students' preferences (e.g., based on their votes) and run ACDA.
- Can be EF- k for any chosen k , while no guarantee on efficiency; but students' welfare can be good in practice assuming their preferences are similar.

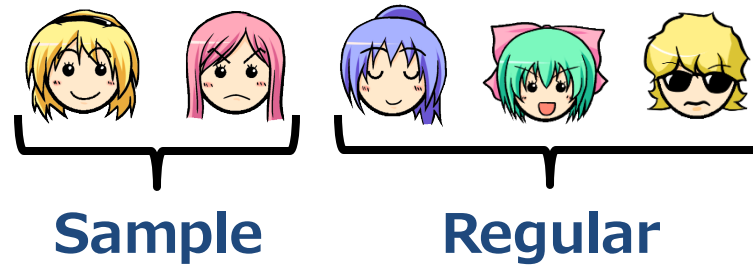
Finding an optimal master-list in SD*

- A student: a vertex
- A directed edge from s to s' : some college prefers s over s'
- An optimal master-list: a serial order among vertices minimizing the number of **reverse** edges **for each vertex**
- A **greedy** algorithm that sequentially chooses a vertex that has the least outgoing edges (and places it to the bottom of the master-list) is **optimal**.



Sample and Deferred Acceptance (SDA)

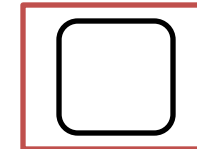
1. Divide the students into two groups.



<Colleges>

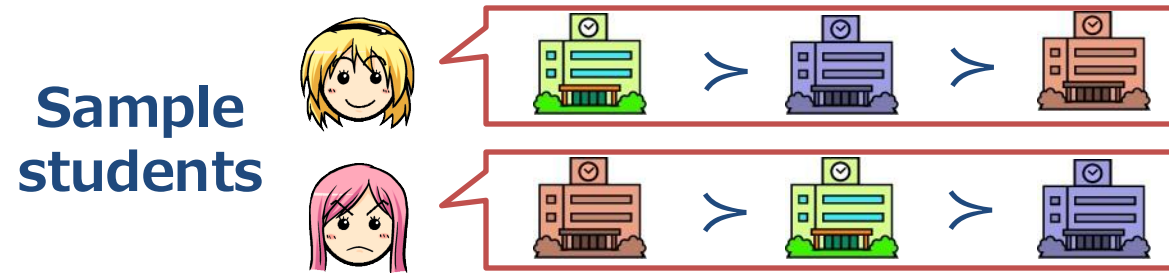


<Resources>



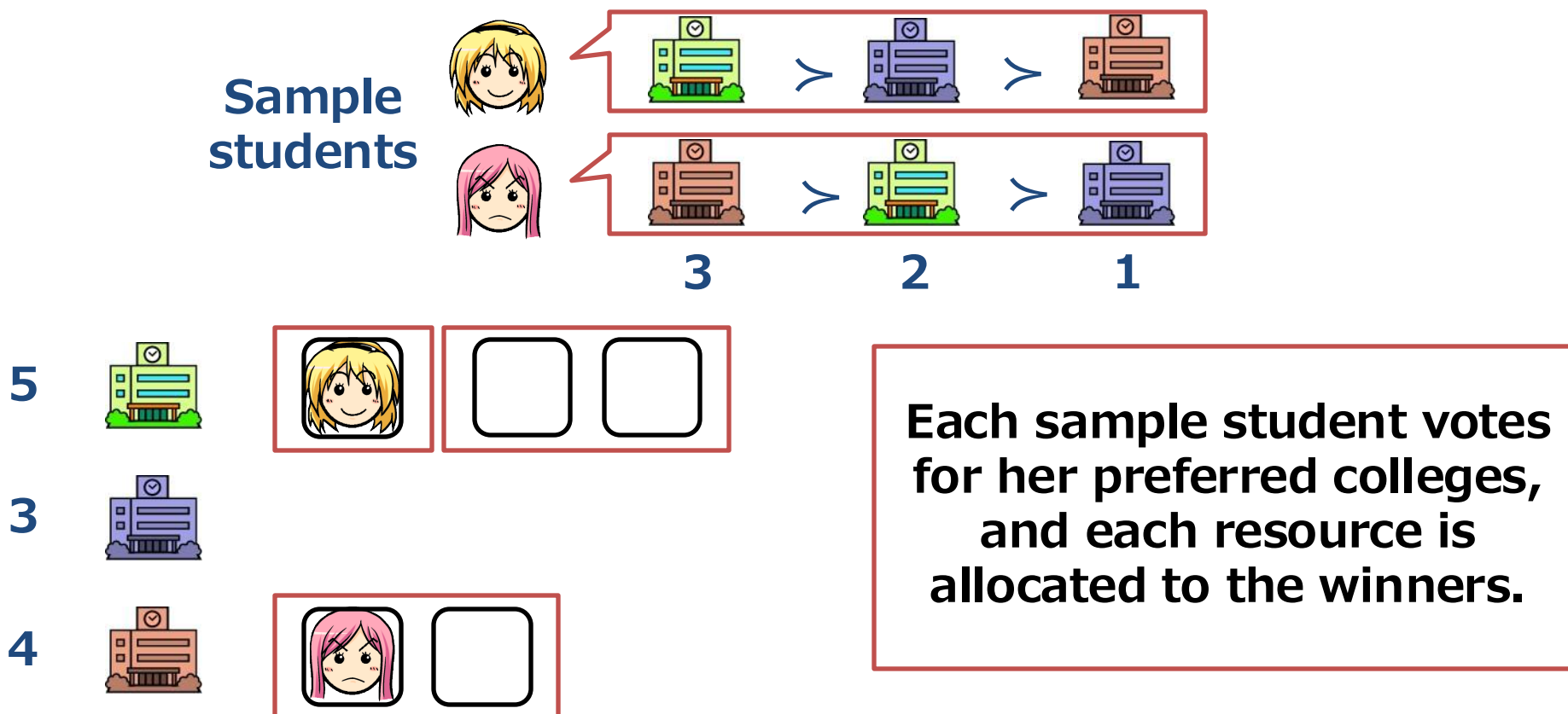
Sample and Deferred Acceptance (SDA)

2.Run **SD** for **sample** students.



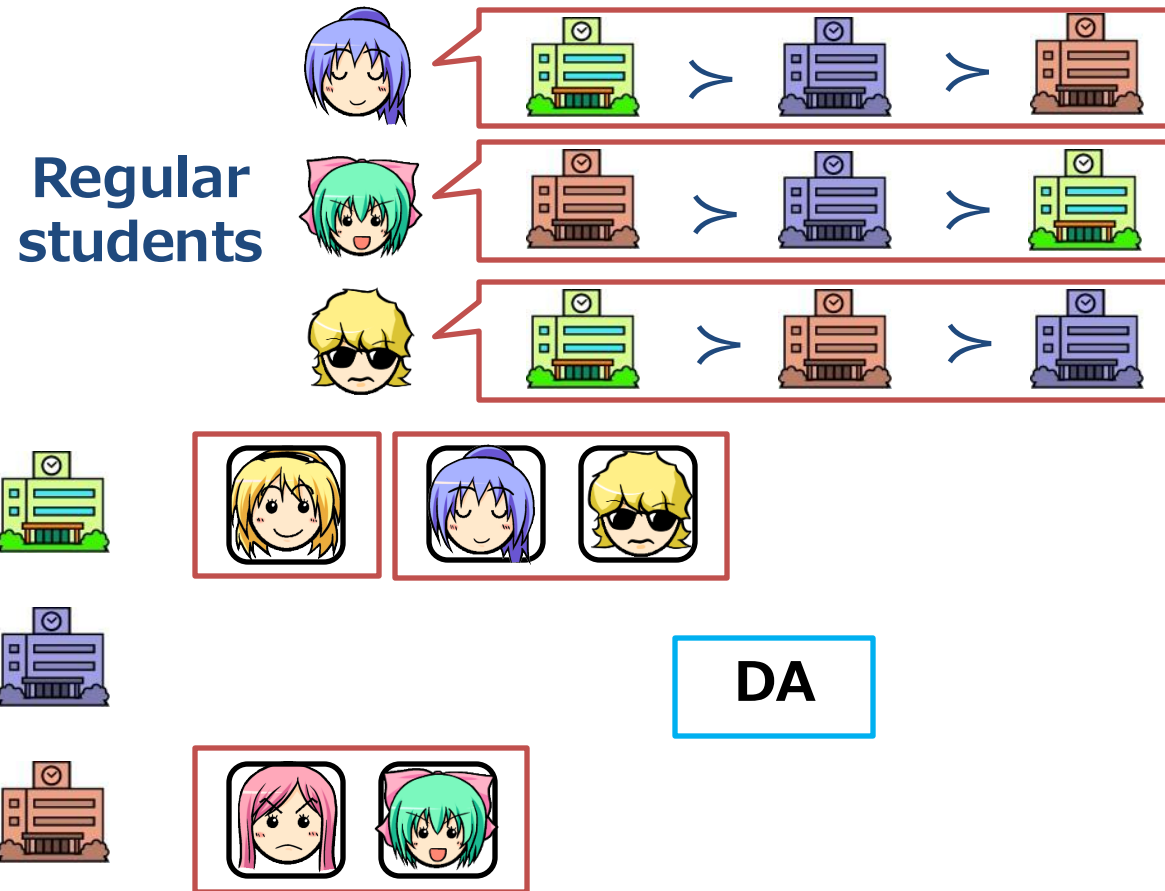
Sample and Deferred Acceptance (SDA)

3. Determine the maximum quotas of colleges (i.e., **resource allocation**) based on the preferences of **sample** students.



Sample and Deferred Acceptance (SDA)

4. Run **ACDA** for regular students.



Properties of SDA

- It is **strategyproof**.
 - For sample students, since SD is strategyproof, they have no incentive for misreporting their preferences.
 - For regular students, since DA is strategyproof and colleges' maximum quotas are determined independently from their preferences, they have no incentive for misreporting.
- It is EF- k assuming there are k sample students.
 - Each sample student only envies other sample students.
 - Each regular student only envies sample students.

Take home message

- Although two-sided matching has a long research history, many research issues remain, in particular, imposing distributional constraints for the outcome.
- Various distributional constraints have been considered. We can analyze distributional constraints based on a simple abstract model.
- There exist mechanisms that can handle some general classes of constraints.

ERATO Kojima Market Design Project

- A big 5-year project supported Japan Science and Technology Agency (JST) has just started.
- PI: Fuhito Kojima (University of Tokyo)
- Co-PI (Computer Science): Makoto Yokoo
- Co-PI (Discrete Math): Akihisa Tamura
- ...



We are looking for postdocs!

References

- Cho, S., Koshimura, M., Mandal, P., Yahiro, K. and Yokoo, M.: Impossibility of weakly stable and strategy-proof mechanism, *Economics Letters*, 217, (2022).
- Fragiadakis, D., Iwasaki, A., Troyan, P., Ueda, S., Yokoo, M.: Strategyproof matching with minimum quotas. *ACM Transactions on Economics and Computation* 4(1), 6 (2015).
- Gale, D., Shapley, L.: College admissions and the stability of marriage. *The American Mathematical Monthly* 69(1), (1962).
- Goto, M., Kurata, R., Kojima, F., Kurata, R., Tamura, A, Yokoo, M.: Designing Matching Mechanisms under General Distributional Constraints, *American Economic Journal: Microeconomics*, 9 (2):226-62, (2017).
- Hatfield, J. W., Milgrom, P. R.: Matching with Contracts, *American Economic Review* 95 (4) (2005).
- Kamada, Y., Kojima, F.: Efficient matching under distributional constraints: Theory and applications. *American Economic Review* 105 (1), (2015).
- Kojima, F., Tamura, A., Yokoo, M.: Designing matching mechanisms under constraints: An approach from discrete convex analysis, *Journal of Economic Theory*, 176 (2018)
- Murota, K.: Discrete convex analysis, *Mathematical Programming* 83 (1998).

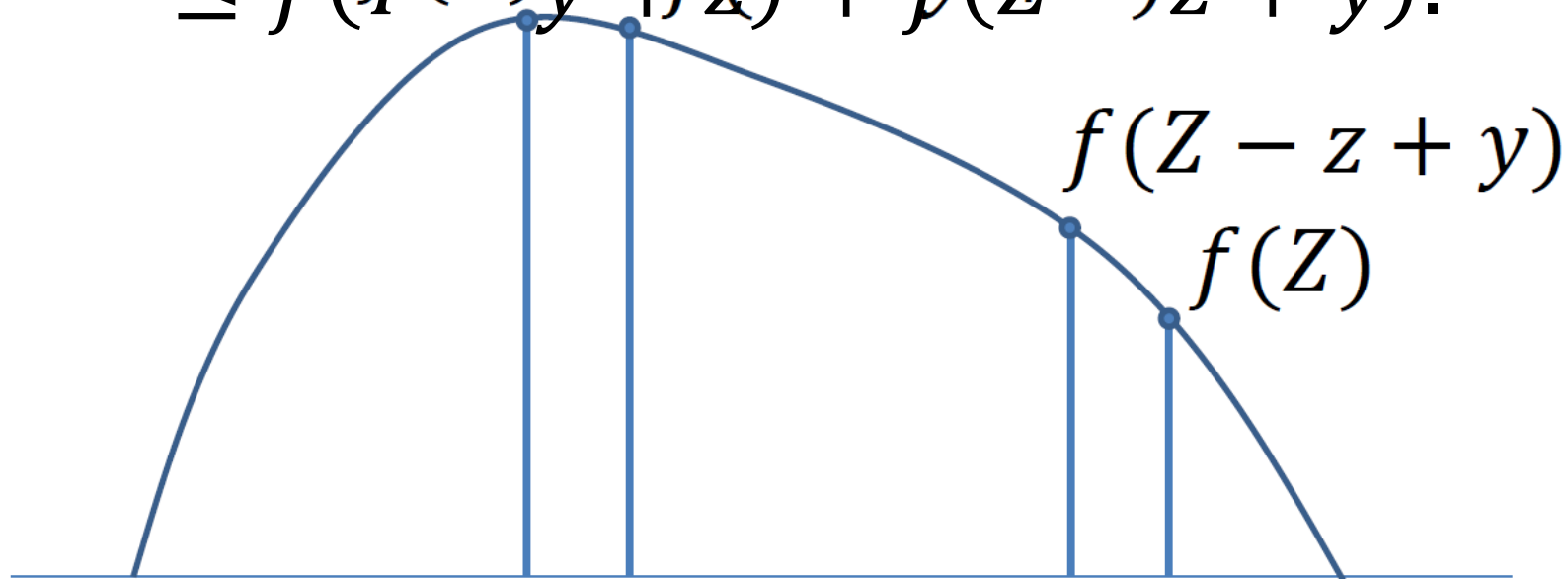


M $\sqcup$ -concavity

(Murota & Shioura, MathOR-1999)

- We say f is M $\sqcup$ -concave if:
for all $Y, Z \subseteq X$, and $y \in Y \setminus Z$, there
exists $z \in (Z \setminus Y) \cup \{\emptyset\}$ s.t.

$$f(Y) + f(Z) \leq f(Y \cup \{y\}) + f(Z \setminus \{z\} \cup \{y\}).$$



When Generalized DA works (I)?

Theorem (Kojima, Tamura, Yokoo, JET-2018):

When f is an $M^\sharp$ -convex function, then generalized DA is (i) strategyproof for students, (ii) obtains the student-optimal matching among all HM-stable matchings, and (iii) its time complexity is $O(T(f) \cdot |X|^2)$, where $T(f)$ is the required time to calculate f .

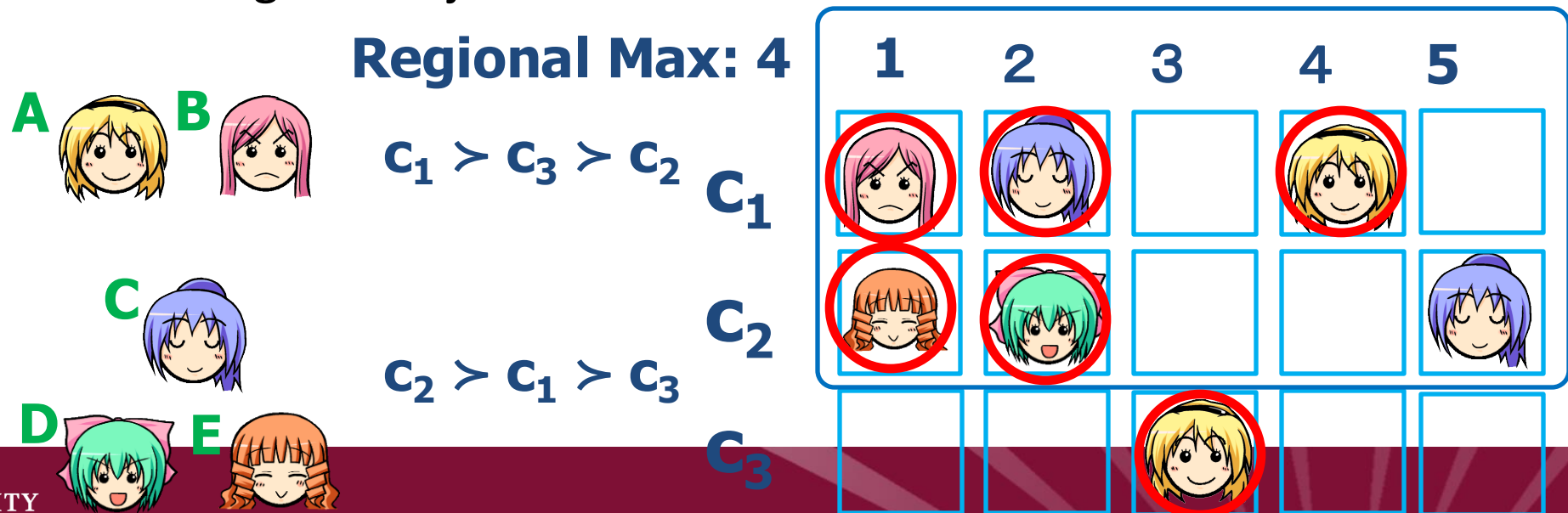
$M \models$ -convex set

and weak nonwastefulness

- Assume $\tilde{f}$ is defined as a weighted sum and each contract has a positive weight.
- Generalized DA obtains a HM-stable matching.
- Assume Y obtained by Generalized DA is not weakly nonwasteful, i.e., for student s , $c \succ_s Y_s$, and $Y + (s, c)$ is feasible.
- Then, $(s, c) \in Ch_S(Y + (s, c))$ and $(s, c) \in Ch_C(Y + (s, c)) = Y + (s, c)$ holds. This contradicts with the fact that Y is HM-stable.

Existing mechanism for regional maximum quotas: Flexible DA (Kamada & Kojima AER-2015)

- Apply the following procedure in each step in DA.
- Assume a round-robin ordering among colleges within a region is defined (e.g., $c_1 \rightarrow c_2 \rightarrow c_1 \rightarrow \dots$).
- According to this ordering, each college deferred-accepts one student as long as doing so does not violate individual/regional maximum quotas. Otherwise, all remaining students applying to the college are rejected.



Characteristics of Flexible DA

- strategyproof
- Fair
- Can be wasteful.
- More flexible than ACDA (more students can be assigned to a popular college).
- Equivalent to generalized DA where $\tilde{f}$ is defined such that having a more balanced assignment among colleges within a region is better (the overall evaluation function f becomes M -concave).

Constraints represented as M $\bowtie$ -convex set (II)

Type-specific maximum quotas:

- We assume v has $|C| \times |T|$ elements. $\hat{f}(v) = 0$ iff $\sum_{c \in C} \sum_{t \in T} v_{c,t} \leq n$, $\sum_{t \in T} v_{c,t} \leq q_c$ holds for each $c \in C$, and $v_{c,t} \leq q_{c,t}$ holds for each $t \in T$ and $c \in C$.
- Let us assume college c is divided into type-specific sub-colleges $c_{t_1}, \dots, c_{t_k}$. Then, this model is equivalent to regional maximum quotas; these type-specific sub-colleges belong to the same region c with regional maximum quota q_c .

Effect of minimum quotas

- Fairness and nonwastefulness can be incompatible.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$		$\succ_{s_1}$	$\succ_{s_2}$
	s_2	s_2	s_1		c_2	c_3
	s_1	s_1	s_2		c_3	c_2
p_c	1	0	0		c_1	c_1
q_c	1	1	1			

Type-specific maximum quotas (Fragiadakis & Troyan, TE-2017)

In addition to the standard model,

- Students are divided into several types: $T = \{t_1, \dots, t_k\}$.
- Each college c has type-specific maximum quota $q_{c,t}$ for each type t .
- We assume v has $|C| \times |T|$ elements. $\hat{f}(v) = 0$ iff $\sum_{c \in C} \sum_{t \in T} v_{c,t} \leq n$, $\sum_{t \in T} v_{c,t} \leq q_c$ holds for each $c \in C$, and $v_{c,t} \leq q_{c,t}$ holds for each $t \in T$ and $c \in C$.

Motivation:

- A type of student may represent her race, gender, or any socioeconomic status. We aim to achieve the diversity of types for each college by restricting the number of students for each type.

Regional maximum quotas

- Fairness and weak nonwastefulness are compatible.
- $Y = \{(s_2, c_1)\}$ does not satisfy nonwastefulness but it satisfies weak nonwastefulness.

	$\succ_{c_1}$	$\succ_{c_2}$		$\succ_{s_1}$	$\succ_{s_2}$
	s_2	s_1		c_1	c_2
	s_1	s_2		c_2	c_1
q_c	1	1			

Regional maximum quota: 1